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## ON HILBERT'S 17<sup>TH</sup> PROBLEM

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### **ABSTRACT**

Hilbert's 17<sup>th</sup> problem, if every positive semi-definite polynomial can be expressed as a sum of squares of rational functions. It was proven affirmatively by Emil Artin. If any polynomial in  $n$  variables that is always non-negative ( $p(x) \geq 0 \forall x \in \mathbb{R}$ ) can be written as a sum of squares of rational functions. One major extension involves the number of squares needed. Artin and Pfister showed that only  $2^n$  squares are needed and later results have sought to improve this number.

### **INTRODUCTION:**

On August 8, 1900, David Hilber in his famous address at the International Congress of Mathematics in Paris proposed twenty-three problems as sign posts for the twentieth century mathematics. Hilbert's 17<sup>th</sup> problem is about the resolution of positive semi-definite functions in  $n$ -variables over reals as sums of squares. Artin in 1927 solved the problem affirmatively using the theory of formally real fields developed by Artin and Schreier in this paper. The classical proof of 17<sup>th</sup> problem of Hilber.

A basic algebraic property of the field of real numbers is that the only relation of the form  $\sum \alpha_i^2 = 0$  which can hold in this field are the trivial ones:  $0^2 + 0^2 + \dots + 0^2 = 0$ . This observation led Artin and Schreier to call any field having this property formally real. Any such field can be ordered and on the other hand any ordered field is formally real. Of central interest in this theory

are the real closed field which are the formally real fields maximal under algebraic extension.

**Definition:** A field  $F$  is said to be formally real if  $-1$  is not a sum of squares in  $F$ .

If  $F$  is formally real then  $ChF = 0$ . For if  $ChF > 0$ ,  $ChF = p$  for some prime  $p$  and so  $p \cdot 1 = 1 + 1 + 1 + \dots + 1 = 0$  in  $F$ . Thus  $-1$  is a sum of squares and so  $F$  is not formally real. However, the converse is not true.  $F$  may not be formally real even when  $ChF = 0$ . For example  $\mathbb{C}$  is not formally real because  $-1 = i^2$  although  $Ch\mathbb{C} = 0$ . If  $F$  is formally real then so is  $F(x)$ . Thus  $R(x_1, \dots, x_n)$  is formally real. Let  $\sigma(F)$  denote the set of elements of  $F$  which can be expressed as a sum of finite number of squares:

$$\sigma(F) = \{x \in F / x = \sum x_i^2, x_i \in F\}.$$

**Proposition 1:** For any field  $F$ .

- (i)  $\sigma(F)$  is closed under addition.
- (ii)  $\sigma(F) - \{0\}$  is a group under multiplication.
- (iii) If  $F$  is not formally real and  $ChF \neq 2$ , then  $\sigma(F) = F$ .

**Proof:** Proof of (i) is clear if  $0 \neq x = x_1^2 + x_2^2 + \dots + x_n^2$  in  $F$ .

$$\text{Then } \frac{1}{x} = \frac{x}{x^2} = \frac{x_1^2 + x_2^2 + \dots + x_n^2}{x^2} = \left(\frac{x_1}{x}\right)^2 + \left(\frac{x_2}{x}\right)^2 + \dots + \left(\frac{x_n}{x}\right)^2 \in \sigma(F) - \{0\}.$$

Also, for  $x, y \in \sigma(F) - \{0\}$ , we have

$$xy = (x_1^2 + x_2^2 + \dots + x_n^2)(y_1^2 + y_2^2 + \dots + y_m^2) = \sum_{i=1}^n \sum_{j=1}^m (x_i y_j)^2 \in \sigma(F) - \{0\}. \text{ This proves (ii).}$$

Since  $F$  is not formally real, we can write  $-1$  as  $-1 = x_1^2 + x_2^2 + \dots + x_n^2, x_i \in F$ .

Let  $x \in F$ , then  $x$  can be written as

$$x = \left(x + \frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2 = \left(x + \frac{1}{4}\right)^2 + \left(x - \frac{1}{4}\right)^2 \{x_1^2 + \dots + x_n^2\} \text{ and so } x \in \sigma(F).$$

Hence  $F = \sigma(F)$ . This proves (iii).

**Definition:** An ordering of a field  $F$  is the assignment of a set  $p \subset F$  called the positive set of the ordering which possesses the following properties:

- (i)  $0 \neq P$  (ii) if  $0 \neq x \in F$ , then either  $x \in P$  or  $-x \in P$ . (iii)  $P$  is closed under addition and multiplication (i.e.,  $a, b \in P \Rightarrow a + b \in P, ab \in P$ )

We can define the relation " $<$ " in  $F$  with  $x < y \Leftrightarrow y - x \in P$ , then ' $<$ ' satisfies the following conditions:

- (i) For  $x, y \in F$  one and only one of the following is true:  $x = y, x < y, x > y$ .
- (ii)  $x \geq 0, y \geq 0 \Rightarrow xy \geq 0, x + y \geq 0$ .
- (iii)  $x^2 \geq 0$  for all  $x \in F$ .

If  $P$  is an ordering over  $F$  and if  $F_0$  is a subfield of  $F$ , then we may order  $F_0$  by declaring  $P_0 = P \cap F_0$  as the set of positive elements in  $F_0$ . The ordering on  $F_0$  is called the ordering induced by the ordering on  $F$ .

**Proposition 2:** Let  $P$  be an ordering over  $F$ .

Then (i)  $\sigma(F) - \{0\} \subseteq P$ .

- (ii)  $F$  is formally real (in particular  $ChF = 0$ ).
- (iii) If  $P' \subseteq F$  is another ordering on  $F$ . then  $P \subseteq P'$ .

**Proof:**

- (i) Since  $P$  is closed with respect to addition, it is sufficient to show that  $x^2 \in P$  for each  $x \neq 0 \in F$ , If  $x \in P$  then  $(x)(x) = x^2 \in P$ . If  $-x \in P$ , then  $x^2 = (-x)(-x) \in P. P \subseteq P$ .
- (ii) If  $-1$  is a sum of squares, then  $-1 \in \sigma(F) - \{0\}$

Hence from (i)  $-1 \in P$ . But  $1 = 1^2 \in P$  and thus  $1 - 1 = 0 \in P$ , a

contraction.

- (ii) Suppose  $x \in P'$  and  $x \notin P$ . Then  $-x \notin P \subseteq P'$   
 Now  $x \in P', -x \in P'$  implies  $0 = x - x \in P'$  a contradiction.  
 Hence  $P = P'$ .

**Basic Lemma:** If  $F$  is formally real but the quadratic extension  $F(\sqrt{a})$  is not formally real, then,  $-a \in \sigma(F)$ .

**Proof:** Since  $F(\sqrt{a})$  is not formally real,  $-1$  can be written as  $-1 =$

$\sum_{i=1}^n (\alpha_i + \beta_i \sqrt{a})^2$ , where  $\alpha_i, \beta_i \in F$

$$\begin{aligned} &= \sum_{i=1}^n \alpha_i^2 + \sum_{i=1}^n \beta_i^2 a + \sum_{i=1}^n 2\alpha_i \beta_i \sqrt{a} \\ &= \sum_{i=1}^n \alpha_i^2 + a \sum_{i=1}^n \beta_i^2 + 2\sqrt{a} \sum_{i=1}^n \alpha_i \beta_i \end{aligned}$$

Since  $\{1, \sqrt{a}\}$  is a basis for  $F(\sqrt{a})$  over  $F$ , we have  $-1 = \sum_{i=1}^n \alpha_i^2 + a \sum_{i=1}^n \beta_i^2$ .

Now  $\sum_{i=1}^n \beta_i^2 = 0$  implies  $-1 = \sum_{i=1}^n \alpha_i^2$  which is contradiction.

Since  $F$  is formally real. Hence  $\sum_{i=1}^n \beta_i^2 \neq 0$

Thus  $-a = \frac{1 + \sum_{i=1}^n \alpha_i^2}{\sum_{i=1}^n \beta_i^2} = \frac{(1 + \sum_{i=1}^n \alpha_i^2)(\sum_{i=1}^n \beta_i^2)}{(\sum_{i=1}^n \beta_i^2)^2}$  and so  $-a \in \sigma(F) - \{0\}$ .

**Definition:** A field  $F$  is called Pythagorean if sum any two squares in  $F$  is always a square. In a Pythagorean field  $F$  any sum of squares is a square and so  $\sigma(F) = F^2 = \{x^2/x \in F\}$ .

Note that  $\mathbb{R}$  and  $\mathbb{C}$  are Pythagorean fields.  $\mathbb{Q}$  is not Pythagorean.

**Definition:** A field  $F$  is said to be real closed if  $F$  is formally real but no proper algebraic extension of  $F$  is formally real. Clearly  $\mathbb{R}$  is a real closed field.

**Proposition 3:** If  $F$  is real closed then  $F$  is Pythagorean.

**Proof:** Assume that  $x^2 + y^2$  is not a square. Then  $F(\sqrt{x^2 + y^2})$  is a quadratic extension of  $F$ .

Since  $F$  is real closed  $F(\sqrt{x^2 + y^2})$  is not formally real.

Hence by the Basic Lemma  $-(x^2 + y^2)$  is a sum of square in  $F$ , say

$$-(x^2 + y^2) = \sum_{i=1}^n z_i^2, \quad z_i \in F$$

i.e.,  $x^2 + y^2 + \sum_{i=1}^n z_i^2 = 0$  which implies

$-1 = \frac{y^2 + \sum_{i=1}^n z_i^2}{x^2} = \left(\frac{y}{x}\right)^2 + \sum_{i=1}^n \left(\frac{z_i}{x}\right)^2$  a contradiction, since  $F$  is formally real.

**Theorem 1:** Let  $F$  be a real closed field. Then

(1) For any  $0 \neq x \in F$  either  $x \in F^{\cdot 2}$  or  $-x \in F^{\cdot 2}$  (where  $F^{\cdot 2}$  denotes the set of all non-zero squares in  $F$ ).

(2)  $F$  has a unique ordering for which the set of positive elements in  $F^{\cdot 2}$ .

**Proof:**

(1) Suppose  $x \notin F^{\cdot 2}$ . Hence  $F(\sqrt{x})$  is a quadratic extension of  $F$ . Since  $F$  real closed  $F(\sqrt{x})$  is not formally real and so by the Basic lemma  $-x$  is a sum of squares. Since a real closed field is Pythagorean, any sum of squares is a square in  $F$ . Thus  $-x$  is a square in  $F$ . i.e.,  $-x \in F^{\cdot 2}$ .

(2) We show that  $P = F^{\cdot 2}$  satisfies the axioms Proposition 1. (i), (ii) and (iii) clearly  $0 \notin F^{\cdot 2}$  and so proposition 1. (i) is clear. For  $0 \neq x \in F$ ,  $x \in F^{\cdot 2}$  or  $-x \in F^{\cdot 2}$  from (1). Hence proposition 1(ii) follows. For  $x^2, y^2 \in F^{\cdot 2}$ ,  $x^2 + y^2 = z^2$ . Since  $F$  is Pythagorean and so  $F^{\cdot 2}$  is closed with respect to addition. Also  $(x^2)(y^2) = (xy)^2 \in F^{\cdot 2}$  and so  $F^{\cdot 2}$  is closed with respect to multiplication. Hence proposition 1(iii) is satisfied because  $F$  has an ordering for which  $F^{\cdot 2}$  is the positive set. Let  $P$  be any ordering of  $F$ . Then  $F^{\cdot 2} \subseteq P$ . Since  $F^{\cdot 2}$  is an ordering of  $F$  this implies from proposition  $F^{\cdot 2} = P$ . Hence  $F$  has a unique ordering for which  $F^{\cdot 2}$  is the positive set.

**Theorem 2:** Let  $F$  be any formally real field and  $\bar{F}$  its algebraic closure then there exist a real closed field  $\Delta$  between  $F$  and  $\bar{F}$ .

**Proof:** Consider the collection  $\zeta$  of all formally real such field of  $\bar{F}$  containing  $F$ . if  $\{F_\alpha\}$  is an inductive family (relative to inclusion) of such fields. Then  $F_\alpha = \bigcup_\alpha F_\alpha$

clearly belongs to the same collection  $\zeta$  by Zorn's Lemma, there exists  $\Delta \in \zeta$  which is a maxima member of  $\zeta$  relative to inclusion,  $\Delta$  is clearly real closed.

**Corollary 1:**  $F$  is formally real if and only if  $F$  possesses at least one ordering.

**Proof:** If  $F$  is an ordered field then clearly from proposition 2(ii).  $F$  is formally real.

Conversely, assume that  $F$  is formally real. By the proposition 2 there exists an algebraic extension  $\Delta$  containing  $F$  which is real closed. The unique ordering on  $\Delta$  induces an ordering on  $F$  and hence  $F$  can be ordered.

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