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ON MINIMAX MULTI OBJECTIVE FRACTIONAL OPTIMALITY CONDITIONS WITH CONVEXITY

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ABSTRACT

Under different forms of convexity conditions, sufficient Kuhn] Tucker conditions and three dual models are presented for the minimax fractional programming.

1. INTRODUCTION

The necessary and sufficient conditions for generalized minimax pro-gramming were first developed by Schmitendorf [16]. Convexity assumptions in the sufficient optimality of [16] and also employed the optimality conditions to construct several dual problems which involve pseudo convex and quasiconvex functions, and they also derived weak and strong duality theorems.

Recently, Hanson [7] introduced the concept of a differentiable invex function which is a generalization of the convex function, and he proved the Kuhn Tucker sufficient optimality theorem and the duality theorem for the nonlinear programming problem involving differentiable invex functions. In this paper, Some definitions and notations are given in section 2.In section 2, We also establish the sufficient conditions for the minimax multi objective fractional programming problem with pseudo convexity. When the sufficient conditions are utilized, one parametric dual problem and two parametric-free dual problems may be formulated and duality results are presented in section 3,4and5

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout, this paper, let R^n be the n-dimensional Euclidean space and R_q^n be its non-negative orthant. We now consider the following minimax fractional problem

$$\text{(FP) Minimize } F(x) = \sup_{y \in Y} \frac{f_i(x,y)}{h_i(x,y)}$$

$$\text{subject to } g_j(x) \le 0$$

Where

(a) Y is a component subset of R^m,

(b) $f_i(.,.)$: $R^n \times R^m \rightarrow R$ is a differentiable function with $f_i(x,y) \ge 0$,

(c) $h_i(., .)$: $R^n \times R^m \rightarrow R$ is a differentiable function with $h_i(x,y) > 0$,

(d) $g_j(.)$: $R^n \rightarrow R^p$ is a differentiable function.

We let

$$J = \{1,2,..,p\}$$

$$J(x) = \{ j \in J \mid g_j(x) = 0\}$$

$$Y(x) = \left\{ y \in Y \left| \frac{f_i(x,y)}{h_i(x,y)} = \sup_{Z \in Y} \frac{f_i(x,y)}{h_i(x,y)} \right. \right\}$$

$$K = \{(s,t,\bar{y}) \in N \times R_+^s \times R^{ms} \mid 1 \le s \le n+1, t = (t_1, \dots, t_s) \in R_+^s$$

with $\sum_{i=1}^{s} t_i = 1$ and $\bar{y} = (y_1, \dots, y_s)$ with $y_i \in Y(x), i = 1, \dots, s$ }.

In [5], Chandra and Kumar derived the following necessary conditions for optimality of (P):

THEROM 2.1 (Necessary Conditions)[5]. Let x^* be a (P)-optimal solution and $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then there exist $(s^*, t^*, \bar{y}) \in K, v^* \in R,$ and $\mu^* \in R_+^p$ such that

$$\sum_{i=1}^{s^*} t_i^* \{\nabla f_i(x^*, y_i) - v^* \nabla h_i(x^*, y_i)\} + \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \qquad (2)$$

$$f_i(x^*, y_i) - v^* h_i(x^*, y_i) = 0, i = 1, \dots, s, \qquad (3)$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \qquad (4)$$

$$\mu^* \in R_+^p, t_i^* \geq 0, \sum_{i=1}^{s^*} t_i^* = 1, y_i \in Y(x^*), i = 1,..., s^*. \tag{5}$$

In order the relax the convexity assumption in Theorem 2.1, we impose the following generalized convexity introduced by Hanson [7]:

Definition 2. 1. Let $\phi: X \to R$ (where $X \subseteq R^n$) be a differential function

(a) The function ϕ is said to be convex at x_0 if there exists for all $x \in X$,

$$\phi(x) - \phi(x_0) \geq (x - x_0)^T \nabla\phi(x_0).$$

Further, ϕ is said to be strictly convex at x_0 if there exists for all $x \in X$, $x \neq x_0$,

$$\phi(x) - \phi(x_0) > (x - x_0)^T \nabla\phi(x_0).$$

(b) The function ϕ is said to be pseudo convex at x_0 if there exists for all $x \in X$,

$$(x - x_0)^T \nabla\phi(x_0) \geq 0 \Rightarrow \phi(x) \geq \phi(x_0)$$

or equivalently

$$\phi(x) < \phi(x_0) \Rightarrow (x - x_0)^T \nabla\phi(x_0) < 0.$$

Further , ϕ is said to be strictly pseudo convex at x_0 if there exists for all $x \in X$, $x \neq x_0$,

$$(x - x_0)^T \nabla\phi(x_0) \geq 0 \Rightarrow \phi(x) > \phi(x_0)$$

Or equivalently

$$\phi(x) \leq \phi(x_0) \Rightarrow (x - x_0)^T \nabla\phi(x_0) < 0.$$

(c) The function ϕ is said to be quasi convex at x_0 if there exists for all $x \in X$,

$$\phi(x) \leq \phi(x_0) \Rightarrow (x - x_0)^T \nabla\phi(x_0) \leq 0.$$

Theorem 2.2 (Sufficient conditions).

Assume that $(x^*, \mu^*, v^*, s^*, t^*, \bar{y})$ satisfy relations (2)-(5). If

$\sum_{i=1}^{s} t_i^* (f_i(., y_i) - v^* h_i(., y_i))$ is a pseudo-convex function at x^*, then x^* is an optimal solution of (P).

Proof: Let x be any feasible solution of (P). From (1) and (4), we have

$$\sum_{j=1}^{p} \mu_j^* g_j(x) \le 0 = \sum_{j=1}^{p} \mu_j^* g_j(x^*).$$

Using the quasi-convexity of $\sum_{j=1}^{p} \mu_j^* g_j(.) = 0$ at x^*, we have

$$(x - x^*)^T \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) \le 0. \qquad (6)$$

Relation (2) along with (6) yields

$$(x - x^*)^T \nabla \sum_{i=1}^{s^*} t_i^* (f_i(x^*, y_i) - v^* h_i(x^*, y_i)) \ge 0. \qquad (7)$$

Along with the facts that $\sum_{i=1}^{s^*} t_i^* (f_i(., y_i) - v^* h_i(., y_i))$ is a pseudo-convex at x^*,

we get from (7) and (3)

$$0 = \sum_{i=1}^{s^*} t_i^* (f_i(x^*, y_i) - v^* h_i(x^*, y_i)) \le \sum_{i=1}^{s^*} t_i^* (f_i(x, y_i) - v^* h_i(x, y_i)) .$$

Therefore , there exists a certain i_0 ,such that

$$0 \le f_i(x, y_{i_0}) - v^* h_i(x, y_{i_0}) .$$

It follows that

$$\sup_{y\in Y}\frac{f_i(x,y)}{h_i(x,y)} \geq \frac{f_i(x,y_{i_0})}{h_i(x,y_{i_0})} \geq v^* = \sup_{y\in Y}\frac{f_i(x^*,y)}{h_i(x^*,y)}.$$

Thus the proof of theorem is complete.

Remark 2.1. In theorem 2.2, If $\sum_{i=1}^{s^*} t_i^* \,(f_i(.\ ,y_i) - v^* h_i(.,y_i))$ is a quasi-convex function at x^* and $\sum_{j=1}^{p}\mu_j^* g_j(.)$ is a strictly pseudo-convex function at x^*, then the Theorem 2.2 also holds.

3. DUALITY MODEL

Making the use of the optimality conditions of the preceding section, we introduce a dual (DI) to the minimax problem (P) as follows:

(DI) $\max_{(s,t,\bar{y})\in} \sup_{(z,\mu,v)\in H_1(s,t,\bar{y})} v$, where $H_1(s,t,\bar{y})$ denotes the set of all triplets

$(z, \mu, v) \in R^n \times R_+^p \times R_+$ satisfying

$$\sum_{i=1}^{s} t_i \{\nabla f_i(z\ ,y_i) - v\ \nabla h_i(z,y_i)\} + \nabla \sum_{j=1}^{p}\mu_j g_j(z) = 0, \tag{8}$$

$$\sum_{i=1}^{s} t_i \ (f_i(z\ ,y_i) - v\ h_i(z,y_i)) \geq 0, \tag{9}$$

$$\sum_{j=1}^{p}\mu_j g_j(z) \geq 0, \tag{10}$$

$$(s,t,\bar{y}) \in K.$$

If for a triplet $(s,t,\bar{y})\in K$ the set $H_1(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$.

THEOREM 3.1 (Weak Duality). Let x and $(z,\mu,v,s,t,\bar{y})$ be (P)-feasible and be (DI)-feasible, respectively, and assume that $\sum_{i=1}^{s} t_i \ (f_i(.\ ,y_i) - v\ h_i(.,y_i))$ is a pseudo-convex function at z, and $\sum_{j=1}^{p}\mu_j g_j(.)$ is a quasi-convex function at z. Then $\sup_{y\in Y}(f_i(x,y)/h_i(x,y)) \geq v.$

Proof: By both the feasibility of x and (10), we have

$$\sum_{j=1}^{p}\mu_j g_j(x) \le 0 \le \sum_{j=1}^{p}\mu_j g_j(z).$$

Using the quasi-convex of $\sum_{j=1}^{p}\mu_j g_j(.)$ at z, we have

$$(x-z)^T \nabla \sum_{j=1}^{p}\mu_j g_j(z) \le 0. \qquad (11)$$

consequently, (8) and (11) yield

$$(x-z)^T \nabla \sum_{i=1}^{s} t_i \,(f_i(z\ ,y_i) - v\ h_i(z,y_i)) \ge 0. \qquad \textbf{(12)}$$

On account of pseudo-convexity of $\sum_{i=1}^{s} t_i \,(f_i(.,y_i) - v\ h_i(.,y_i))$ at z, we get from (12) and (9)

$$0 \le \sum_{i=1}^{s} t_i \,(f_i(z\ ,y_i) - v\ h_i(z,y_i)) \le \sum_{i=1}^{s} t_i \,(f_i(x,y_i) - v\ h_i(x,y_i)).$$

Therefore, there exists a certain i_0 **, such that**

$$0 \le f_i(x,y_{i_0}) - v\ h_i(x,y_{i_0}).$$

It follows that

$$\sup_{y\in Y}\frac{f_i(x,y)}{h_i(x,y)} \ge \frac{f_i(x,y_{i_0})}{h_i(x,y_{i_0})} \ge v.$$

Thus, the proof of the theorem is complete.

Remark 3.1. In Theorem 3.1, if $\sum_{i=1}^{s} t_i \,(f_i(.,y_i) - v\ h_i(.,y_i))$ **is a quasi-convex function at** z and $\sum_{j=1}^{p}\mu_j g_j(.)$ is a strictly pseudo-convex function at z, then Theorem 3.1 also holds.

THEOREM 3.2 (Strong Duality). Assume that x^* is a (P)-optimal solution and $\nabla g_i(x),\ j \in J(x^*)$ **is linearly independent. If in addition the hypothesis of Theorem 3.1 holds for all (DI)-feasible points** $(z,\mu,v,s,t,\bar{y})$, **then the two problems (P) and (DI) have the same external values.**

Proof: By theorem 2.1 there exists $(s^*, t^*, \bar{y}) \in K, (x^*, \mu^*, v^*) \in H_1(s^*, t^*, \bar{y})$ **such that** $(x^*, \mu^*, v^*, s^*, t^*, \bar{y})$ is a (DI)-optimal solution.

Since $v^* = f_i(x^*, y_i)/h_i(x^*, y_i),$ optimality of this feasible solution for (DI) follows from Theorem 3.1.

THEOREM 3.3 (Strict Converse Duality). Let $\bar{x}$ **and** $(z, \mu, v, s, t, \bar{y})$ be optimal solutions of (P) and (DI), respectively, and assume that $\sum_{i=1}^{s} t_i (f_i(., y_i) - v\ h_i(., y_i))$ **is a pseudo-convex function at** z for all $(s, t, \bar{y}) \in K, (x, \mu, v) \in H_1(s, t, \bar{y}), \sum_{j=1}^{p} \mu_j g_j(.)$ **is a quasi-convex function at** z and, $\nabla g_i(\bar{x}), j \in J(\bar{x})$ **is linearly independent. Then** $\bar{x} = z$**; that is,** z is a (P)-optimal solution and $\sup_{y \in Y}(f_i(z, y)/h_i(z, y)) = v.$

Proof: We shall assume that $\bar{x} \neq z$ and reach a contradiction. From Theorem 3.2 we know that

$$\sup_{y \in Y} \frac{f_i(\bar{x}, y)}{h_i(\bar{x}, y)} = v. \tag{13}$$

By both the feasibility of $\bar{x}$ and (10), we have

$$\sum_{j=1}^{p} \mu_j g_j(\bar{x}) \leq 0 \leq \sum_{j=1}^{p} \mu_j g_j(z).$$

Using the quasi-convex of $\sum_{j=1}^{p} \mu_j g_j(.)$,we get from the inequality above

$$(\bar{x} - z)^T \nabla \sum_{j=1}^{p} \mu_j g_j(z) \leq 0. \tag{14}$$

Consequently, (8) and (14) yield

$$(\bar{x} - z)^T \nabla \sum_{i=1}^{s} t_i (f_i(z\ \ , y_i) - v\ h_i(z, y_i)) \geq 0. \tag{15}$$

On account of **strict pseudo-convexity of** $\sum_{i=1}^{s} t_i (f_i(., y_i) - v\ h_i(., y_i))$ **at** z, we get

From (15) and (9)

$$0 \le \sum_{i=1}^{s} t_i \,(f_i(z\ \ , y_i) - v\ \ h_i(z, y_i)) < \sum_{i=1}^{s} t_i \,(f_i(x, y_i) - v\ \ h_i(x, y_i)).$$

Therefore, there exists a certain i_0 **, such that**

$$0 < f_i(x, y_{i_0}) - v\ \ h_i(x, y_{i_0}).$$

It follows that

$$\sup_{y \in Y} \frac{f_i(x, y)}{h_i(x, y)} \ge \frac{f_i(x, y_{i_0})}{h_i(x, y_{i_0})} > v = \sup_{y \in Y} \frac{f_i(z, y)}{h_i(z, y)}.$$

Which contradicts (13), and the proof is complete.

4. DUALITY MODEL II

In order to discuss another dual model for (P), we first state another version of Theorem 2.1. This is done by replacing the parameter v^* with $f_i(x^*, y_i)/h_i(x^*, y_i)$ and by rewriting the multiplier functions associated with the inequality constraints. The result of Theorem 2.1 can be stated as follows.

THEOEM 4.1.If x^* is an optimal solution of the problem (P) and $\nabla g_i(x), j \in J(x^*)$ is linearly independent, then there exists $(s^*, t^*, \bar{y}) \in K$ and $\mu^* \in R_+^p$ such that

$$\sum_{i=1}^{s^*} t_i^* \{ h(x^*, y_i) \nabla f_i(x^*, y_i) - f_i(x^*, y_i) \nabla h_i(x^*, y_i) \} + \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0,$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0,$$

$$\mu^* \in R_+^p, t_i^* \ge 0, \sum_{i=1}^{s^*} t_i^* = 1, y_i \in Y(x^*), i = 1,..., s^*.$$

Now we introduce a dual (DII) to the minimax problem (P)

(DII) $\max_{(s,t,\bar{y})\in K} \sup_{(z,\mu)\in H_2(s,t,\bar{y})} f(z)$, where $H_2(s,t,\bar{y})$ denotes the set of $(z,\mu)\in R^m \times R_+^p$ satisfying

$$\sum_{i=1}^{s} t_i \{h(z,y_i)\nabla f_i(z,y_i) - f_i(z,y_i)\nabla h_i(z,y_i)\} + \nabla\sum_{j=1}^{p}\mu_j g_j(z) = 0, \quad (16)$$

$$\sum_{j=1}^{p}\mu_j g_j(z) \geq 0, \quad (17)$$

$y_i \in Y(z), i = 1,\ldots, s,$

and

$$F(z) = \sup_{y\in Y}\frac{f_i(z,y)}{h_i(z,y)}. \quad (18)$$

If the set $H_2(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$. Throughout this section, we simply denote $\psi(.)$ as

$$\sum_{i=1}^{s} t_i \{h(z,y_i) f_i(i,y_i) - f_i(.,y_i) h_i(.,y_i)\}.$$

THEOEREM 4.2 (Weak Duality). Let x be (P)-feasible and $(z,\mu,v,s,t,\bar{y})$ be (DII)-feasible, respectively, and assume that $\psi_1(.)$ is a pseudo-convex function at z, and $\sum_{j=1}^{p}\mu_j g_j(.)$ is a quasi-convex function at z. Then $\sup_{y\in Y}(f_i(x\ ,y_i)/h_i(x\ ,y_i)) \geq F(z)$.

Proof: By both the feasibility of x and (17), we have

$$\sum_{j=1}^{p}\mu_j g_j(x) \leq 0 \leq \sum_{j=1}^{p}\mu_j g_j(z).$$

Using the quasi-convexity $\sum_{j=1}^{p}\mu_j g_j(.)$ at z, we have

$$(x-z)^T \nabla\sum_{j=1}^{p}\mu_j g_j(z) \leq 0. \quad (19)$$

Consequently, (16) and (19) yield

$$(x-z)^T \nabla \psi_1(z) \geq 0. \quad \textbf{(20)}$$

On account of pseudo-convexity of $\psi_1(.)$ at z, we get from (20)

$$0 = \psi_1(z) \leq \psi_1(.x).$$

Therefore, there exists a certain i_0, such that

$$0 \leq h_i(z, y_{i_0}) f_i(x, y_i) - f_i(z, y_i) h_i(x, y_{i_0}).$$

It follows that

$$\sup_{y \in Y} \frac{f_i(x,y)}{h_i(x,y)} \geq \frac{f_i(x, y_{i_0})}{h_i(x, y_{i_0})} \geq \frac{f_i(z, y_{i_0})}{h_i(z, y_{i_0})}. \quad (21)$$

Since $y_{i_0} \in Y(z)$, we have

$$F(z) = \frac{f_i(z, y_{i_0})}{h_i(z, y_{i_0})}. \quad (22)$$

Thus the proof of the theorem is complete.

Remark 4.1. In theorem 4.2, if $\psi_1(.)$ is a quasi-convex function at z, and $\sum_{j=1}^{p} \mu_j g_j(.)$ is a strictly pseudo-convex function at z, then Theorem 4.1 also holds.

Similar to the proof of Theorems 3.2, 3.3, we can establish Theorems 4.3, 4.4. Therefore, we simply state them.

THEOREM 4.3 (Strong Duality). If x^* is a (P)-optimal solution and $\nabla g_i(x), j \in J(x^*)$ is linearly independent, and if in addition the hypothesis of Theorem 4.2 holds for all (DII)-feasible points $(z, \mu, s, t, \bar{y})$, then the two problems (P) and (DII) have the same external values.

THEOREM 4. 4 (Strict Converse Duality). Let $\bar{x}$ and $(z,\mu,v,s,t,\bar{y})$ be optimal solutions of (P) and (DII)-feasible points $(z,\mu,v,s,t,\bar{y})$, $\psi_1(.)$ is strictly pseudo-convex at z, and $\sum_{j=1}^{p}\mu_j g_j(.)$ is quasi-convex at z. Then $\bar{x}=z$: that is, z is a (P)-optimal solution.

5. DUALITY MODEL III

Based on (2) and (3), we obtain

$$\nabla\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)-\frac{f_i(x^*,y_i)}{h_i(x^*,y_i)}\nabla\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)+\nabla\sum_{j=1}^{p}\mu_j^* g_j(x^*)=0, \text{ for all } i=1,\dots,s^*.$$

Multiplying the above equations respectively by $t_i^*(h_i(x^*,y_i)$, $i=1,\dots,s^*$, and adding them up, we have

$$\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)\nabla\left[\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)+\sum_{j=1}^{p}\mu_j^* g_j(x^*)\right]-\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)\nabla\left[\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right]=0.$$

The above equation, together with (4), implies

$$\nabla\left(\frac{\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)+\sum_{j=1}^{p}\mu_j^* g_j(x^*)}{\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)}\right)$$

$$=\left(\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right)^{-2}\times\left(\left[\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right]\nabla\left[\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)+\sum_{j=1}^{p}\mu_j^* g_j(x^*)\right]-\left[\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)+\sum_{j=1}^{p}\mu_j^* g_j(x^*)\right]\nabla\left[\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right]\right)$$

$$=\left(\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right)^{-2}\times\left(\left[\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right]\nabla\left[\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)+\sum_{j=1}^{p}\mu_j^* g_j(x^*)\right]\right.$$

$$\left.-\left[\sum_{i=1}^{s^*}t_i^* f_i(x^*,y_i)\right]\nabla\left[\sum_{i=1}^{s^*}t_i^* h_i(x^*,y_i)\right]\right)$$

$$=0.$$

Consequently, the result of Theorem 2.1 can be stated as follows.

THEOREM 5.1. If x^* is an optimal solution of the problem (P) and $\nabla g_i(x), j \in J(x^*)$ is linearly independent, then there exist $(s^*, t^*, \bar{y}) \in K$ and $\mu^* \in R_+^p$ such that

$$\nabla\left(\frac{\sum_{i=1}^{s^*} t_i^* f_i(x^*, y_i) + \sum_{j=1}^{p} \mu_j^* g_j(x^*)}{\sum_{i=1}^{s^*} t_i^* h_i(x^*, y_i)}\right) = 0,$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0,$$

$$\mu^* \in R_+^p, t_i^* \geq 0, \sum_{i=1}^{s^*} t_i^* = 1, y_i \in Y(x^*), i = 1,..., s^*.$$

We shall continue our discussion of the parameter-free duality model for (P) in this section by showing that the following variant of (DIII) is aiso the dual problem for (P):

(DIII) $\max_{(s,t,\bar{y})\in K} \sup_{(z,\mu)\in H_2(s,t,\bar{y})} \dfrac{\left(\sum_{i=1}^{s} t_i f_i(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)\right)}{\left(\sum_{i=1}^{s} t_i^* h_i(z, y_i)\right)}$ where $H_3(s,t,\bar{y})$ denotes the set $(z,\mu) \in R^m \times R_+^p$ satisfying

$$\nabla\left(\frac{\sum_{i=1}^{s} t_i f_i(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i^* h_i(z, y_i)}\right) = 0. \qquad (23)$$

If the set $H_3(s,t,\bar{y})$ is empty, then we define the supremum over it to be $-\infty$. Throughout this section, we simply denote $\psi_1(.)$ as

$$\sum_{i=1}^{s} t_i h_i(z, y_i)\left[\sum_{i=1}^{s} t_i f_i(.,y_i) + \sum_{j=1}^{p} \mu_j g_j(.)\right] - \left[\sum_{i=1}^{s} t_i f_i(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)\right]\sum_{i=1}^{s} t_i h_i(.,y_i).$$

We shall state weak duality, strong, and strict converse duality theorems for (P)-(DIII).

THEOREM 5.2 (Weak Duality). Let x be (P)-feasible and $(z,\mu,s,t,\bar{y})$ be (DIII)-feasible, and assume that $\psi_2(.)$ is a pseudo-convex function at z . Then

$$\sup_{y\in Y}\frac{f_i(x,y)}{h_i(x,y)}\geq\frac{\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i h_i(z,y_i)}.$$

Proof: From (23), we have

$$\nabla\psi_2(z)=0. \tag{24}$$

By means of contradiction, we suppose that

$$\sup_{y\in Y}\frac{f_i(x,y)}{h_i(x,y)}<\frac{\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i^{*}h_i(z,y_i)}.$$

Thus we have an inequality

$$f_i(x,y)\sum_{i=1}^{s}t_i h_i(z,y_i)<h_i(x,y)\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right] \text{ For all } y\in Y.$$

Furthermore, this above inequality implies

$$\left[\sum_{i=1}^{s}t_i f_i(x,y_i)\right]\left[\sum_{i=1}^{s}t_i h_i(z,y_i)\right]<\left[\sum_{i=1}^{s}t_i h_i(x,y_i)\right]\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right].$$

Hence, we have another inequality

$$\left[\sum_{i=1}^{s}t_i f_i(x,y_i)+\sum_{j=1}^{p}\mu_j g_j(x)\right]\left[\sum_{i=1}^{s}t_i h_i(z,y_i)\right]-\left[\sum_{i=1}^{s}t_i h_i(x,y_i)\right]\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right]$$

$$<\sum_{j=1}^{p}\mu_j g_j(x)\sum_{i=1}^{s}t_i h_i(z,y_i).$$

Using the fact that $\sum_{i=1}^{s} t_i h_i(z, y_i) > 0, \sum_{j=1}^{p} \mu_j g_j(x) \le 0,$ and the latest inequality, we have

$$\psi_2(x) < 0 = \psi_2(z).$$

Using the fact that $\psi_2(.)$ is a pseudo-convex function at z, we have

$$(x - z)^T \nabla \psi_2(z) < 0. \qquad (25)$$

But (24) and (25) are not compatible. Thus, the proof is complete.

Remark 5.1. If we add the constraint $\sum_{i=1}^{s} t_i f_i(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z) \ge 0$ to the dual problem (DIII), and if $f_i(. , y_i), -h_i(., y_i), g_j(.)$ are convex functions, and $(x - z)^T = x - z$ we can reduce Theorem 5.2 to Theorem 3.2, of Chandra and Kumar [5].

As a consequence of Theorems 5.1 and 5.2, we obtain the following strong duality theorem:

THEOREM 5.3 (Strong Duality). If x^* is a (P)-optimal solution and $\nabla g_i(x), j \in J(x^*)$ is linearly independent, and if in addition the hypothesis of Theorem 5.2 holds for all (DIII)-feasible points $(z, \mu, s, t, \bar{y})$, then the two problems (P) and (DIII) have the same extremal values.

THEOREM 5.4 (Strict Converse Duality). Let $\bar{x}$ and $(z, \mu, s, t, \bar{y})$ be optimal solutions of (P) and (DIII), respectively, and assume that $\psi_2(.)$ is a strictly pseudo-convex function at x for all $(s, t, \bar{y}) \in K, (z, \mu) \in H_3(s, t, \bar{y}),$ and $\nabla g_i(x), j \in J(x^*)$ is linearly independent. Then $\bar{x} = z;$ that is, z is a (P)-optimal solution.

Proof: We shall assume that $\bar{x} \ne z;$ and reach a contradiction. From Theorem 5.3 we know that

$$\sup_{y \in Y} \frac{f_i(\bar{x}, y)}{h_i(\bar{x}, y)} = \frac{\sum_{i=1}^{s} t_i f_i(z, y_i) + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i h_i(z, y_i)}.$$

Hence we have

$$f_i(\bar{x},y)\sum_{i=1}^{s}t_i h_i(z,y_i) \le h_i(\bar{x},y)\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right] \text{ For all } y\in Y.$$

Furthermore, this above inequality implies

$$\left[\sum_{i=1}^{s}t_i f_i(\bar{x},y_i)\right]\left[\sum_{i=1}^{s}t_i h_i(z,y_i)\right] \le \left[\sum_{i=1}^{s}t_i h_i(\bar{x},y_i)\right]\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right].$$

Hence, we have another inequality

$$\left[\sum_{i=1}^{s}t_i f_i(\bar{x},y_i)+\sum_{j=1}^{p}\mu_j g_j(\bar{x})\right]\left[\sum_{i=1}^{s}t_i h_i(z,y_i)\right]-\left[\sum_{i=1}^{s}t_i h_i(\bar{x},y_i)\right]\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right]$$

$$\le \sum_{j=1}^{p}\mu_j g_j(\bar{x})\sum_{i=1}^{s}t_i h_i(z,y_i).$$

Using the fact that $\sum_{i=1}^{s}t_i h_i(z,y_i)>0, \sum_{j=1}^{p}\mu_j g_j(\bar{x})\le 0,$ and the latest inequality, we get

$$\psi_2(\bar{x})\le 0=\psi_2(z).$$

With the strict pseudo-convexity of $\psi_2(.)$, we have $(x-z)^T\nabla\psi_2(z)<0.$ that is,

$$(\bar{x}-z)^T\left\{\sum_{i=1}^{s}t_i h_i(z,y_i)\nabla\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right]-\left[\sum_{i=1}^{s}t_i f_i(z,y_i)+\sum_{j=1}^{p}\mu_j g_j(z)\right]\left[\nabla\sum_{i=1}^{s}t_i h_i(z,y_i)\right]\right\}<0. \quad (26)$$

But (23) and (26) are not compatible. This completes the proof.

REFERENCES

1. C. R. Bector and B. L. Bhatia, sufficient optimally conditions and duality for a minimax problem, Utilitas Maths . (27) (1985), 229-347.
2. C. R. Bector, S. Chandra and I. Husain, second order duality for a minimax programming problem, Opsearch 28 (1991), 257-263.
3. C. R. Bector, S. Chandra and V. Kumar, duality for a minimax programming involving V-convex functions, optimization 30 (1993). 93-103.

4. D. Bhatia and P. Kumar, multi objective control problem with generalized convexity, J. Math. Anal. Appl. 189 (1995), 676-692.
5. S. Chandra and V. Kumar, duality in fractional minimax programming J. Austral. Math. Soc. Ser. A 58 (1995), 376-386.
6. R. R. Egudo and M. A. Hason, multiobjective duality with convexity, J. Math. Anal. Appl. 126 (1987), 469-477.
7. M. A. Hanson, On sufficient of the Kuhn-tucker conditions, J. Math. Anal. Appl. 80 (1981), 545-550.
8. M. A. Hanson and B. Mond, Further generalizations of convexity in mathematical programming, J. Inform. Optim. Sci. 3 (1983), 25-32.
9. V. Jeyakumar and B. Mond , On generalized convex mathematical programming, J. Austral. Math. Soc. Ser. B 34 (1992), 43-53.
10. R. N. Kunal, S. K. Suneja, and M. K. Srivastava, Optimality criteria and duality in multiple-objective optimization involving generalized convexity, J. Optim. Theory Appl. 80 (1994), 465-482.
11. J. C. Liu, Optimality and duality for generalized fractional programming involving nonsmooth pseudo-convex functions, J. Math. Anal. Appl. 202 (1996), 667-685.
12. J. C. Liu, Optimality and duality for multiobjective fractional programming involving nonsmooth pseudo-convex functions, Optimization 37 (1996), 27-39.
13. J. C. Liu, Sufficiency criteria and duality in complex nonlinear programming involving pseudo-convex functions, Optimization 39 (1997), 123-135.
14. J. C. Liu, C. S. Wu, and R. L. Sheu, Duality for fractional monomax programming, Optimazation, 41 (1997), 117-133.
15. J. C. Liu, C. S. Wu, on minimax fractional optimality conditions with (F,ρ) -convexity, J. Math. Anal. Appl. (to appear).
16. W. E. Schmitendorf, necessary conditions and sufficient conditions for static minimax problems, J. Math. Anal. Appl. 57 (1977), 683-693.